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An Implication of the Existence of Competitive Equilibrium

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1 Introduction

The purpose of this note is pedagogic. One of the most fundamental propositions in economic theory is that a competitive equilibrium exists. What is entirely non-intuitive about this proposition is the fact that the set of assumptions, which allows a wide variety of decision makers to do their own thing, is mutually consistent. That consumers and producers, each engaged in purely selfish pursuits while believing that one is unable to control prices to one's own advantage, are able to make plans which are mutually compatible, seems surprising. Fifty two years ago, in the December 1952 Meetings

[1]The note was prepared at the insistence of Satish Jain and I cannot absolve him of responsibility for the contents. I am greatly indebted to a referee for very helpful comments. **Email:** anjan@mail.jnu.ac.in

of the Econometric Society, two papers[2] were presented which provided a demonstration of this proposition. Apart from laying the corner stone for not only economic theory, but also for much of economic policy that was to follow, the papers provided a bench mark for rigour in theorizing which have been difficult to match. Fascinating as this literature may be, we are not concerned directly with it[3].

We are concerned with one aspect of this literature and it is this: the demonstrations of existence of competitive equilibrium since 1952, follow directly from a class of mathematical theorems known as fixed point theorems[4]. Is this a coincidence or merely an example of the tool-fetish that theorists are some times charged with ? Uzawa (1962), in an elegant contribution, showed that proving the existence of competitive equilibrium was equivalent to a particular fixed point theorem and hence it was not surprising that such a powerful result was needed.

As a careful reading of the Uzawa paper will make clear, existence of competitive equilibrium, when excess demands are **functions** of prices, are continuous over the unit simplex[5] and satisfy Walras Law imply that **any** continuous function from the unit simplex to itself admits a fixed point (Brouwer's Fixed Point Theorem)[6]. The problem however is that when one considers excess demands to be **functions**, the underlying preferences and technology

[2]The papers were by McKenzie, and by Arrow and Debreu.

[3]For excellent reviews see, Debreu (1982), McKenzie (1981).

[4]If a function $f(x)$ is defined over some set A with values also in the same set A, a point $\bar{x}$ in A is said to be a fixed point of the function $f(.)$ if $f(\bar{x}) = \bar{x}$. Fixed point Theorems provide conditions under which such points exist.

[5]For a definition of such a set, see below.

[6]The Uzawa argument has been reproduced in Mukherji (2002), p.69-70 since the original is generally difficult to locate.

sets need to be restricted so that decision making leads to unique responses. One such restriction is the assumption of **strict convexity**; this, in turn, implies that the excess demand functions are defined only when all prices are positive[7]. Thus we cannot expect the excess demand functions to be even well defined over the entire unit simplex and hence continuity of excess demand functions on the unit simplex seems to be infeasible. It is because of this reason that a more acceptable requirement is that excess demand be continuous only in the **interior** of the unit simplex; in such a situation, we may need an alternative tool, the Kakutani Fixed Point Theorem[8]. And the question then is whether this version of the Fixed Point Theorem is still equivalent to the existence of competitive equilibria in an appropriate setting.

More specifically, we shall consider the existence of free disposal equilibria[9] and we shall show that this existence result implies the Kakutani Fixed Point Theorem and is in fact, equivalent to it.

The survey by Debreu (1982) does indeed look into a similar issue and provides a geometric argument which establishes this proposition. Debreu attributes the argument to Uzawa, but it does appear that the ingenious and simple Uzawa (1962) construction is lost sight of. We provide an argument which is the original Uzawa (1962) contribution adapted to the changed set-up; at the same time, it may be seen to be making the Debreu argument explicit. Alternatively, the main result of this note may be seen as an extension

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[7]Strict convexity ensures, for example, that indifference curves do not have straight line segments; thus when some price is zero, there may not be a tangency between a budget line and an indifference curve and hence, the notion of demand is not well defined.

[8]See, for instance, Mukherji (2002), p. 81, Section 4.6.

[9]The Debreu-Gale-Nikaido Lemma, Gale (1955), Nikaido (1956), Debreu (1956), implies this result.

of the argument reported in Border (1985) to demonstrate the equivalence between the existence of free disposal equilibria when excess demand is single valued and Brouwer's Fixed Point Theorem[10]. In fact, the monograph by Border (1985), contains many insights (Chapters 9 and 21). into the connections between the fixed point theorems in mathematics and existence results in economics; while the connection presented below does not find a specific mention in the monograph, it may be deduced in a straightforward manner.

2 Definitions

An economy is described by a **finite number** of competitive decision makers or economic agents; some agents decide on what to buy and sell to meet their own needs; these are the consumers; there are other agents who decide on what to produce given the level of technology that they may have access to; they too decide on what to buy and sell. They are competitive in the sense that they cannot control prices to their own advantage and they take the prices as data while formulating their plans to buy and sell. Whereas the first set of agents decide on their actions so as to maximize their level of well being subject to their purchasing power, or utility, (such an utility maximizing choice is called "demand"), the second set of agents decide on the basis of what constitutes maximum profits (a profit maximizing choice is called "supply"). The decisions of all such agents, taken together, contribute to the notion of **excess demand**, which is aggregate demand minus aggregate supply. An equilibrium for this economy is a configuration of **prices** at which excess demand is zero; actually, we shall be concerned with a some what

[10]Sections 9.5, 9.13, Border (1985).

weaker notion of a free disposal equilibrium, which we introduce below.

We shall be concerned with maps $\zeta : \Re^n_+ \to \Re^n$ satisfying the following[11]

i. $W^{12}.\forall p \in \Re^n_+, p.z = 0 \; \forall z \in \zeta(p)$.

ii. $H^{13}.\forall p \in \Re^n_+, z \in \zeta(p) \Rightarrow z \in \zeta(\lambda p) \forall \lambda > 0$.

iii. $Co^{14}.\forall p \in \Re^n_+, \zeta(p)$ is a non-empty convex subset of $\Re^n$.

iv. $Cl^{15}.\forall p \in \Re^n_+$, the map $\zeta(p)$ is closed[16].

Let $S_n = \{p \in \Re^n_+ : \sum_{i=1}^n p_i = 1\}$: the unit simplex in $n-$dimension. By virtue of $H.$ above, we shall consider the map $Z : S_n \to \Re^n$ where $Z(p) = \{z : z \in \zeta(p)\}$; this map satisfies $W., Co.$, and $Cl.$.

Conversely, any map $\phi : S_n \to \Re^n$ satisfying these properties will be called an **excess demand-map**[17]. It may also be noted that beginning with the set up of utility maximizing consumers and profit maximizing firms, excess demands obtained cannot be subjected to any further restrictions than the

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[11]Our notation is standard in such contexts. But for the sake of completeness we provide the following: $\forall$: for all; $\in$: belonging to or in; $\Rightarrow$: implies ; $\Re$ denotes the set of real numbers; $A \times B$: the Cartesian product of the sets A, B consists of ordered pair of elements $(a, b), a \in A, b \in B$; if $A = B$, then $A \times B$ is written as A^2; thus $\Re^n$ denotes the set of n-tuples of real numbers: the n-dimensional Euclidean Space; $\Re^n_+$ denotes the non-negative orthant, or n-tuples of real numbers, each of which is non-negative; $\zeta : A \to B$: for every $a \in A$, we identify a subset $\zeta(a) \in B$, called the image of a under the map ζ ; $\sum$: sum; $\cap$: intersection or the elements common to the sets appearing after the sign; $\emptyset$: the empty set.

[12]Walras Law.

[13]Homogeneity of degree zero in prices.

[14]Convexity of the image sets.

[15]The map has to be a closed map; see the next footnote for a definition.

[16]If $p^s \to \bar{p} \in \Re^n_+, p^s \in \Re^n_+ \forall s, z^s \in \zeta(p^s)$ then $z^s \to \bar{z} \in \zeta(\bar{p})$.

[17]See, for example, Debreu (1982), p. 716 for the definition of excess demand correspondence.

ones we have already imposed[18].

Finally let $\mathcal{E} = \{p \in S_n : Z(p) \cap \Re^n_- \neq \emptyset\}$: the set of free disposal equilibria. The term "free disposal" refers to the fact that $p \in \mathcal{E} \Rightarrow \zeta \leq 0, \zeta \in Z(p)$; further since $W.$ is assumed, $p \in \mathcal{E} \Rightarrow \zeta_i < 0$ only if $p_i = 0$; thus if $p \in \mathcal{E}$ and $p_i > 0$ then $\zeta_i = 0 \forall \zeta \in Z(p)$; thus only "free " goods are allowed to be in excess supply in equilibrium and hence the term "free disposal".

Consider next the following:

- **Free Disposal Existence Theorem (FDET)** For any excess demand map map $Z : S_n \rightarrow \Re^n, \mathcal{E} \neq \emptyset$.
- **Kakutani Fixed Point Theorem (KFPT)** For any map $\psi : S_n \rightarrow S_n$ such that $\psi(p)$ is non-empty, convex and closed, there is $p^\star \in S_n$ such that $p^\star \in \psi(p^\star)$.

3 The Proposition: $FDET \Rightarrow KFPT$

The proof will proceed by beginning with any map $\psi : S_n \rightarrow S_n$ satisfying the hypothesis of KFPT. For any such map, define the map $Z : S_n \rightarrow \Re^n$:

$$Z(p) = \{z : z = y - \frac{p.y}{p.p}p,\ y \in \psi(p)\},\ \forall p \in S_n$$

Notice then that $Z : S_n \rightarrow \Re^n$ and satisfies the assumptions $W., Co.$ and $Cl.$. To see these properties, first of all, $z \in Z(p) \Rightarrow p.z = 0$; next, $z^i \in Z(p), i = 1, 2 \Rightarrow \exists y^i \in \psi(p)$ such that

$$z^i = y^i - \frac{p.y^i}{p.p}p\ , i = 1, 2;$$

[18] See, for instance, Shafer and Sonnenschein (1982).

further, notice that by the hypothesis, $y = \lambda y^1 + (1-\lambda)y^2 \in \psi(p)$ for any λ , $0 \leq \lambda \leq 1$. Thus

$$z = \lambda z^1 + (1-\lambda)z^2 = y - \frac{p.y}{p.p}p \Rightarrow z \in Z(p)$$

which implies Co. Cl. also follows: consider $p^s \in S_n, p^s \to \bar{p} \in S_n$, $z^s \in Z(p^s)$ such that $z^s \to \bar{z}$. Since $z^s \in Z(p^s)$, $\exists y^s \in \psi(p^s)$ such that $z^s = y^s - \frac{p^s.y^s}{p^s.p^s}p^s$. Since $p^s \to \bar{p}$ and $y^s \in \psi(p^s) \in S_n$, it follows that y^s is a bounded sequence and has a limit point $\bar{y} \in S_n$ and from the Cl. property for $\psi(p)$ it follows that $\bar{y} \in \psi(\bar{p})$. Thus it follows that $\bar{z} = \bar{y} - \frac{\bar{p}.\bar{y}}{\bar{p}.\bar{p}}\bar{p} \Rightarrow \bar{z} \in Z(\bar{p})$. This demonstrates that the map Z satisfies Cl.

Consequently, the map Z is an excess demand map. By FDET therefore, the set $\mathcal{E} = \{p \in S_n : Z(p) \cap \Re^n_- \neq \emptyset\} \neq \emptyset$. Consequently, there is $p^\star \in S_n$ such that for some $z^\star \in Z(p^\star), z^\star \leq 0$; i.e., there is

$$y^\star \in \psi(p^\star) \tag{1}$$

and

$$z^\star = y^\star - \frac{p^\star.y^\star}{p^\star.p^\star}p^\star \leq 0$$

Further since $p^\star \in S_n$ and $p^\star.z^\star = 0$ it follows that $z^\star_i = 0$ $\forall i$ such that $p^\star_i > 0$. Thus

$$\forall i \text{ such that } p^\star_i > 0,\ y^\star_i = \frac{p^\star.y^\star}{p^\star.p^\star}p^\star_i \tag{2}$$

for all other values of i, $y^\star_i \leq 0$ and hence $y^\star_i = 0$ since $y^\star \in \psi(p^\star) \subset S_n$. Thus for all i, (2) must hold; in other words, we must have

$$y^\star_i = \frac{p^\star.y^\star}{p^\star.p^\star}p^\star_i \ \forall i \tag{3}$$

Consequently summing over i on both sides of (3) and noting that $y^\star, p^\star \in S_n$, we have:

$$p^\star.y^\star = p^\star.p^\star \tag{4}$$

Using (4) in (3), we have $y_i^\star = p_i^\star \ \forall i$ and consequently it follows from (1) that $p^\star \in \psi(p^\star)$ as claimed by KFPT. This completes the demonstration.

4 Concluding Remarks

That $KFPT \Rightarrow FDET$ is well known[19]; as we have indicated above $FDET \Rightarrow KFPT$ by extending the Uzawa (1962) argument; one may therefore conclude:

$$KFPT \Leftrightarrow FDET.$$

To prove the existence of a competitive equilibrium in any general setting, one must use an appropriate fixed point theorem or an equivalent result.

Finally, it may be of some interest to note the method adopted in Border (1985) to develop several interconnections between results in mathematics and results in economics. It may be seen that finding equilibria for an economy is really identical to finding a solution to a system of equations (or a system of inequalities, in the case of free disposal equilibria); location of fixed points of appropriate maps also amount to the same thing. In the case of inequalities, the problem may be transformed into showing that the intersection of appropriate sets are non-empty; this in turn may be related to the problem of finding maximal elements of a binary relation. A binary relation U on a set K is a subset of $K \times K$; if $(x, y) \in U$, we also write xUy or even $x \in U(y)$. A maximal element of the binary relation U is a point $x \in K$ such that $U(x) = \emptyset$. Thus the set of maximal elements of U may be writ-

[19] See, for instance Debreu (1982), p. 717-718, where it is deduced that the KFPT $\Rightarrow$ Debreu-Gale-Nikaido Lemma and which in turn implies the FDET.

ten as $\cap_{y \in K}\{x \in K : yUx\}^{c}$[20]; by identifying sufficient conditions for this intersection to be non-empty, conditions for existence of solutions to inequalities are also established. The construction of an appropriate binary relation, so that its maximal elements are the solutions sought, is of course central. And given the basic fundamental nature of this argument, interconnections between several results are revealed.

[20]The superscript 'c' refers to the complement of a set; for any two sets A, B, if $B \subset A$ then $B^c = A - B$: the set of elements of A which are not in B.

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