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A Note on Rationalizability of Choice Functions

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In the year 1938 Paul Samuelson published his much celebrated paper which laid the foundation of revealed preference approach to the theory of consumer behavior. The approach was distinctly novel in the sense that it tried to explain the demand function from the individual's choice behavior itself and did not need to assume the existence of some ordinal utility function. Attempts were made to see whether the axiom proposed by him, now popularly known as the weak axiom of revealed preference, ensured the existence of an ordinal utility function that can generate such choices. The weak axiom failed to ensure that but succeeded in provoking a vast pool of literature in the area throughout the last century which enriched the consumer theory as well as the theory of social choice. Houthakker (1950) showed that a modified and

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stronger version of the weak axiom is necessary and sufficient for the existence of an ordinal utility function.

Uzawa (1957) and Arrow (1959) extended the framework of revealed preference to social choice theory. Instead of looking at the demand function of a consumer they became interested in choice functions in general where individuals are choosing a non-empty subset from every set that belongs to a class of non-empty sets. The moment we enter the theory of social choice the question that we were asking also somewhat changes. We no longer are looking for an ordinal utility function but are searching for a preference relation that can generate such a choice function. If we can find a preference relation such that the choice sets are always the sets of best elements according to the same preference relation then the choice function is said to be rationalizable. So, we may say that, rationalizability of a choice function is like approving that the choices are not nonsensical and are indeed made by some rational individual. A long list of axioms have been proposed to this end to certify the sensible nature of a choice function. Arrow (1959) proposed an axiom (commonly known as Arrow's axiom or alternatively, conditions α and β[1] combined) and showed that it is necessary and sufficient for rationalization of a choice function by an ordering if the domain of the choice function contains all non-empty finite subsets. As a matter of fact with such a domain restriction a series of axioms including weak and strong axioms of revealed preference, Houthakker's axiom, weak and strong congruence axioms, Arrow's axiom; they all become equivalent. If, however, we do not impose any domain restriction then it is Houthakker's axiom (which is equivalent to strong con-

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[1] See Sen (1977).

gruence axiom[2]) that serves the same purpose[3]. Attempts were also made to find conditions of rationalizability of choice functions by preference relations that are not orderings. If the domain of a choice function contains all non empty subsets of the set of alternatives then (a) it is rationalizable by a reflexive, connected and quasi-transitive binary relation if and only if it satisfies generalized Condorcet (GC) condition and path independence (PI)[4], (b) it is rationalizable by a reflexive, connected and acyclic binary relation if and only if it satisfies condition α and generalized Condorcet condition[5]. Strictly speaking, these necessary and sufficient conditions ensure the existence of a binary relation such that, for every subset its choice set is the set of its *best elements* according to that same binary relation.

It is quite straight forward that if a choice function is rationalizable then choices are well-behaved. A rather problematic proposition is to say that, for choices to be well-behaved a choice function needs to be rationalizable. If a choice function is not rationalizable in the sense that the chosen elements are not the best elements then is it necessary that the choices made are inconsistent or there may still be some scheme of harmony working underneath? Until towards the end of the last century this question was either overlooked or ignored. Revealed preference approach with its intuitive appeal and superiority over the utility approach became standard and extremely influential - so much so that different axioms of this approach were taken unquestioned as *internal consistency* conditions for a choice function. The idea was that

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[2] See Richter (1966).

[3] See Suzumura (1977) for the proof.

[4] See Plott (1973).

[5] See Blair et al. (1976).

given any choice function one may check for consistency of choices with these conditions without looking at the motivation or objective of the choice. A distinct breakthrough came in 1984 with Sen's Presidential address of the Econometric Society[6] where he argued against *a priori* imposition of internal consistency conditions of choice. He argued that consistency of choice cannot be judged in a context-independent way. In his own words, "there is no 'internal' way - internal to the choice function itself - of determining whether a particular behavior pattern is or is not consistent. The necessity of bringing in something outside choice behavior is the issue" (Sen (1993)).

Taking the cue from Sen's argument we shall be arguing here that choosing a best element is not necessary for a choice function to be consistent. However, if there is no binary relation according to which the best elements are chosen, we can immediately see that such a choice function cannot be rationalized by standard rationalizability conditions. Nevertheless, depending on the objective (which Sen calls *external reference*) someone may reasonably choose an alternative which is not best according to her preference. Standard rationalizability conditions may judge such a choice function as inconsistent, while all this time the person may consistently be choosing the second best elements from the sets of alternatives. Only this consistency is not apparently visible to us. Sen has argued that we can not see that consistency unless we consider the *external reference* or the objective of the chooser. *External reference* is central to the interpretation of a choice function.

Let us consider the following choices:

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[6] See Sen (1993).

$C(\{x,y\}) = \{x\}$

$C(\{x,y,z\}) = \{y\}$

These choices will violate many of the standard rationalizability conditions. Clearly there is no binary relation according to which the best elements were chosen. Does that mean that the person is inconsistent in making choices? Well, that would be too hasty to conclude. Considering the fact that the person has a binary relation $zPyPx$ and is choosing the second best elements from every set we can see that at least in terms of consistency these choices are no less consistent than the case where we always choose the best elements.

The question that one may ask then is why should anybody be at all interested in choosing an inferior element while she can always choose a best element? The situation, although, prima facie may look somewhat perplexing an explanation can be given to show that such a behavior is perfectly reasonable. To illustrate our point we start with Sen's by now classic example. A person may love cakes and her preference may satisfy nonsatiation. Nevertheless, in a party she may not be willing to pick the very largest piece from the tray as she does not want to be taken as greedy. So consistently she picks up the second largest piece of cake. She is a maximizer but her behaviour is guided by a norm: "never pick the largest slice", which she has internalised. Now let us consider the previous choice example. If x, y, and z are sizes of cakes in increasing order then her behavior is perfectly reasonable and the choices made are consistent with the norm she has internalised.

To take another example, suppose in a family there are five brothers of different ages. Their mother might have taught them that when offered pieces of cakes one should leave the bigger pieces for younger brothers. So the eldest brother always leaves the four biggest pieces on the tray and chooses the fifth largest piece. His immediate younger brother chooses the fourth largest piece and leaves three biggest pieces for his brothers and so on. Here every brother is again consistent in his choices although their choices fail to satisfy many of the standard rationalizability conditions.

In both of the previous examples we have seen involvement of some kind of a norm. The norm acts as a constraint to the maximisation problem. It is however not necessary always to be guided by some kind of a norm or social custom to make a choice which does not involve choosing of a best element. We may do so due to purely economic reason. Consider the second-price sealed bid auction. The auctioneer chooses the second highest price from the quotations submitted to him. He does that not because of some norm or custom but because he wants to extract the maximum amount of money from the bidders.

So we see that not choosing the best element from a set is not the same as being unreasonable. A choice function that is not best element rationalizable therefore merits further investigation. Once we recognize this fact it becomes interesting to check whether a choice function is second best element rationalizable or in general k-th best element rationalizable (alternatively k-rationalizable) where k is some positive integer. Such a general condition would help us understand how these rationalizability conditions translate as the value of k changes. It would improve our understanding of choice functions and provide us with a deeper insight.

Best element rationalizability conditions are known to us for a long time. Baigent and Gaertner (1996) had given us 2nd best rationalizability condition for a choice function where one chooses the second best element from a set whenever there is a uniquely best element in the set. Gaertner and Xu (1999) characterized choice functions which are median element rationalizable by a linear ordering. In the present paper we shall see how a k-rationalizability condition can be obtained. We would allow k to assume the value of any positive integer apart from 1[7]. In that way, it is an attempt to get results that are more general than the previous works. Formal proofs of theorems are not presented here[8]; instead we would be relying more on intuitions. The choice behavior that we would be addressing is where one always chooses the k-th best element from a set. She looks for a $(k-1)$-th best element only when k-th best element is not available in the set.

Here we state and intuitively explain the results that have been obtained. For that purpose we introduce some definitions and notation that have been used in proving the results. Let N be the set of positive integers and X be the uni-

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[7] With $k = 1$, we go back to the case of best element rationalizability for which the results are long known.

[8] Formal proofs are given in the paper "Necessary and sufficient condition for a choice function to be k-rationalizable", presented in the Conference on Growth, Inequality and Institutions, November 27-29, 2008, held at Jawaharlal Nehru University, New Delhi. The paper is available at,

http://conf.ciiss.net/index.php/econtheory/cgii/paper/view/1

http://ssrn.com/abstract=1334063

http://www.cigionline.org/cigi/download-nocache/Research/shifting/bricsam/conferen
/paperspres/krationali

versal non-empty finite set of alternatives. Let Σ be the set of all non-empty subsets of X. A *choice function* is defined as a mapping, $C : D \subseteq \Sigma \mapsto \Sigma$ such that $C(A) \subseteq A$ for all $A \in D$. $C(A)$ is called the *choice set* of the set A. Here we assume that the domain of choice function contains all non-empty subsets of X, i.e., $D = \Sigma$. For any binary relation R on a set S, the asymmetric and symmetric parts of R, designated by P and I respectively, are defined as, $(\forall x, y \in S)[[xPy \leftrightarrow xRy \wedge \sim yRx] \wedge [xIy \leftrightarrow xRy \wedge yRx]]$. A binary relation R is said to be an *ordering* if and only if it is reflexive, connected and transitive.

We derive a binary relation R_2 over X as follows:

$$(\forall x, y \in X)[(xR_2y) \leftrightarrow y \in C(\{x, y\})].$$

P_2 and I_2 respectively are the asymmetric and symmetric parts of R_2.

For any set $S \in \Sigma$ we define,

$$B_S = \{x \in S \mid (\forall y \in S)(y \in C(\{x, y\}))\}.$$

For any $x \in X$ we define,

$$P_x = \{y \in X \mid C(\{x, y\}) = \{x\}\}.$$

Clearly B_S is the set of best elements in S with respect to R_2 and P_x is the set of all elements in X that are preferred to x with respect to R_2.

For all $S \in \Sigma$ and for all $x, y \in S$,

$$(x, y) \in P_2|S \leftrightarrow x, y \in S \wedge C(\{x, y\}) = \{y\}.$$

$P_2|S$ is the restriction of P_2 (asymmetric part of R_2) over the set S.

For all $S \in \Sigma$ and for all $x \in S$,

$$S_x^P = \{y \in S \mid (y, x) \in T(P_2|S)\}.$$

$T(P_2|S)$ is the transitive closure of $P_2|S$.

$G(S, R)$ is the set of best elements in S with respect to the binary relation R. We define,

$$\begin{aligned} G_1(S, R) &= G(S, R) && \text{and} \\ G_i(S, R) &= G[S - \bigcup_{j=1}^{i-1} G_j(S, R), R] && \text{where} \quad i \geq 2. \end{aligned}$$

A binary relation R is a *k-rationalization* (kth best element rationalization), $k \in N$, of a choice function C iff,

$$\begin{aligned} C(S) &= G_k(S, R) && \text{if} \quad G_k(S, R) \neq \emptyset \\ &= G_j(S, R) && \text{if} \quad G_j(S, R) \neq \emptyset \ \wedge \ G_{j+1}(S, R) = \emptyset \ \text{ where } 1 < j < k;\ j, k \in N \\ &= G_1(S, R) && \text{if} \quad G_2(S, R) = \emptyset \end{aligned}$$

for all $S \in D$.

For any set $S \subseteq X$, the *order of the set* S, denoted as O_S, is defined as follows:

$$\begin{aligned} O_S &= 0 \quad \text{iff} \quad S = \emptyset \\ &= 1 \quad \text{iff} \quad (\forall x, y \in S)[\{x, y\} \in D \rightarrow C(\{x, y\}) = \{x, y\}] \\ &= n \quad \text{iff otherwise} \end{aligned}$$

where n is the largest value of m such that there exist distinct $z_1, z_2, ..., z_m$ in S and $(\forall i \in \{1, 2, ..., m-1\})[C(\{z_i, z_{i+1}\}) = \{z_{i+1}\}]$.

Intuitively, O_S gives us the largest number of consecutive preference levels present in the set S according to the binary relation P_2. We illustrate the idea by considering the following examples.

Example 1

Let $S = \{a, b, c\}$ and the choice function be $C(\{a, b\}) = \{b\}$, $C(\{b, c\}) = \{c\}$, $C(\{a, c\}) = \{a\}$.

The longest chain of elements joined by P_2 that we may get here such that no element occurs more than once, can be, $(aP_2b \wedge bP_2c)$ or $(bP_2c \wedge cP_2a)$ or $(cP_2a \wedge aP_2b)$. In all cases we have three distinct consecutive preference levels. So the order of the set is 3.

Example 2

Let $S = \{a, b, c, d, e\}$ and the choice function be $C(\{a, b\}) = \{b\}$, $C(\{b, c\}) = \{c\}$, $C(\{d, e\}) = \{e\}$.

The longest chain can be constituted by the elements a, b and c. We have three distinct and consecutive preference levels here with a at the top, b in the middle and c at the bottom. The order of S is 3.

Example 3

Let $S = \{a, b, c, d\}$. Let the choice function C be the following: $C(\{a, b\}) = \{b\}$, $C(\{b, c\}) = \{c\}$, $C(\{c, d\}) = \{d\}$, $C(\{a, c\}) = \{a, c\}$, $C(\{a, d\}) = \{a, d\}$, $C(\{b, d\}) = \{b, d\}$, $C(\{a, b, c, d\}) = \{a, b, c, d\}$.

The longest chain can be constituted by the elements a, b, c, d. The order of S is 4.

We now state the axioms we have used in characterization of a k-rationalizable choice function.

A1: $(\forall S \in \Sigma)[O_S \geq k \rightarrow C(S) = \{x \in S \mid O_{P_x \cap S} = k-1\}]$

A2: $(\forall S, T \in \Sigma)[O_S < k \rightarrow [T \subseteq S \wedge C(S) \cap T \neq \emptyset \rightarrow C(T) = C(S) \cap T]]$

A3: $(\forall S \in \Sigma)[O_S \geq k \rightarrow C(S) = \{x \in S \mid O_{S^P_x} = k-1\}]$

A4: $(\forall S \in \Sigma)[O_S < k \rightarrow C(S) = \{x \in S \mid O_{S^P_x} = O_S - 1\}]$

A5: $(\forall S \in \Sigma)[B_S \cap C(S) \neq \emptyset \ \rightarrow \ (\forall S' \subseteq S)(S' \neq \emptyset \rightarrow C(S') = S')]$

A6: $(\forall S \in \Sigma)[B_S \neq S \ \rightarrow \ C(S) = \{z \in S \,|\, P_z \cap S \neq \emptyset \wedge P_z \cap S \subseteq B_S\}]$

A7: $(\forall S \in \Sigma)(\forall x \in S)[x \notin B_S \ \rightarrow \ B_S \subseteq P_x]$

It has been proved[9] that if the binary relation R_2 is reflexive, connected and acyclic over a set S then the order of the set S ensures the existence of a best element of that order. For example, if the order of some set S is n, $n \in N$, and R_2 over S is reflexive, connected and acyclic then there exists an n-th best element in S.

Let us now have a closer look at the axioms. We start with A1. A1 requires that whenever there is a k-th best element present in a set S, an element will be chosen if and only if the set of all elements in S that are preferred to it is of the order $k-1$. If R_2 is an ordering then A1 ensures that all elements that are k-th best according to R_2, will be chosen and no element will be

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[9] Proof is given in the paper "Necessary and sufficient condition for a choice function to be k-rationalizable".

chosen that is not k-th best whenever there is a k-th best element present in the set. By the definition of k-rationalization with an ordering, in absence of a k-th best element the choice set is essentially the set of worst elements. No wonder that A2 requires Arrow's axiom to hold in all such cases. It has been proved[10] that together A1 and A2 are necessary and sufficient for k-rationalizability of a choice function by an ordering, for all $k \geq 3$. The idea behind the axiom A3 is similar to that of A1. Difference between these two axioms is rather technical. If the binary relation R_2 is acyclic then $P_x \cap S$ is a subset of S_x^P. With a transitive R_2 they become equal. Axiom A4 requires that in the absence of k-th best element in a set, the chosen elements are those for which the order of the set of preferred elements (the preference relation being $T(P_2|S)$) is one less than that of the mother set. Intuitively, according to A4, the set of unchosen preferred elements lacks one preference level than the original set. This missing preference level is constituted by the chosen elements; which makes the chosen elements the set of worst elements in the set. So the spirit behind axiom A2 and A4 are essentially same even though they look quite different. A necessary and sufficient condition for k-rationalization of choice functions by a reflexive, connected and acyclic binary relation has been obtained by combining A3 and A4 when $k \geq 2$.

So the only case that is left now is, characterization of a k-rationalizable choice function for $k = 2$ (which we may call a 2-rationalizable choice function) by an ordering. For that purpose we need three more axioms, namely, A5, A6, A7. Axiom A5 requires that if for any set a best element is chosen then there is no unchosen element in any of its subsets. In a 2-rationalizable

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[10] ibid.

choice function the only case when a best element makes its way to a choice set is when there is no second best element. Notice that in case of a 2-rationalizable choice function with full domain, the absence of a second best element ensures that all elements are best elements. So there is no reason to discriminate between alternatives while making a choice. That makes the requirement of the axiom A1 intuitively clear. If however, not all elements are best then there exists a second best element. Axiom A6 takes care of the situation when a second best element indeed exists. It requires that in such a situation the chosen elements are those for which a preferred element exists and any element preferred to them must be a best element. So it ensures that the chosen elements are second best. Axiom A7 takes care of transitivity of the binary relation. It requires that best elements are preferred to any non-best element. It has been proved[11] that together A5, A6 and A7 are necessary and sufficient for 2-rationalizability of a choice function by an ordering if the domain of the choice function contains all non-empty subsets.

We now have necessary and sufficient conditions for choice functions to be k-rationalizable, for all $k \in N - \{1\}$, by orderings as well as by binary relations that are reflexive, connected and acyclic. Adding them with the already existing best element rationalizability conditions makes the characterization exercise complete for all $k \in N$.

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[11] The proof is given in the paper "Necessary and sufficient condition for a choice function to be 2-rationalizable". The paper is available at,
http://ssrn.com/abstract=1335061

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