Contemporary Issues and Ideas in Social Sciences

December 2007

A Discussion on Continuous Time Growth Models

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1 Introduction

This note discusses an intriguing problem of economic modeling raised by an example due to Tapan Mitra[2]. The example is a special case of the continuous-time Solow model of economic growth (1956).

Example 1 *(Mitra) Consider the Solow model with an intensive production function f defined on the capital labour ratio k. If s is the constant savings ratio and n and δ the constant rates of population growth and depreciation respectively then the basic equation is:*

$$\dot{k}(t) = sf(k(t)) - (n + \delta)k(t) \tag{1}$$

where $\dot{k}(t) = k'(t)$. Suppose that $f(k) = 4\sqrt{k}$ and that $n, \delta = 0$; then

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[2]In a private communication dated March 10, 2006.

$$\dot{k}(t) = 2\sqrt{k(t)} \tag{2}$$

Consider the special case when $k(0) = 0$; then $k(t) = 0, t \geq 0$ is a solution to (2). However,

$$k(t) = t^2, t \geq 0 \tag{3}$$

is also a solution.

In fact, for arbitrary τ, $k(t) = 0$, for $0 \leq t \leq \tau$ and $k(t) = (t - \tau)^2$, for $t \geq \tau$ is a class of solutions.

Remark 1 *Although $f(0) = 0$, growth from a zero initial capital stock is possible. This is puzzling to an economist.*

Remark 2 *Non-uniqueness of the solution to (2) is well-known for this kind of differential equation. The problem stems from the "Inada condition" $f'(0) = +\infty$. That means the marginal productivity of capital is undefined (on the real line) for a zero capital input and consequently the Lipschitz condition that is sufficient for uniqueness of solution to differential equations is violated*[3]*.*

Remark 3 *Although Example 1 is very special, the problem is generic for Cobb-Douglas production functions in the Solow model. Population growth and depreciation may be allowed also. Considering $f(k) = ak^{\alpha}, 0 < \alpha < 1$ the basic growth equation reduces to:*

$$\dot{k}(t) = A(k(t))^{\alpha} - Bk(t) \tag{4}$$

where $A = sa$ and $B = (n + \delta)$. There are two cases to consider:

[3] See e.g., Bellman (1953), p. 69.

(a). $B > 0$. Let $b = (1 - a)B$. It can be verified that

$$k(t) = [\{A(1 - e^{-bt})/B\} + (k(0))^{1-\alpha}e^{-bt}]^{\frac{1}{1-\alpha}} \tag{5}$$

is a solution and that for the special case $k(0) = 0$,

$$k(t) = \{A(1 - e^{-bt})/B\}^{\frac{1}{1-\alpha}} \tag{6}$$

is a solution[4]*.*

(b). $B = 0$. In this case it can be seen that for $k(0) > 0$ we have

$$k(t) = [A(1 - \alpha)t + (k(0))^{1-\alpha}]^{\frac{1}{1-\alpha}} \tag{7}$$

as the solution while for the case $k(0) = 0$, a solution is:

$$k(t) = [A(1 - \alpha)t]^{\frac{1}{1-\alpha}} \tag{8}$$

Remark 4 *One would have thought that there were two steady state equilibria in the Solow model, with the origin $k = 0$ as an 'unstable' steady state. However it may not be accurate to characterize the origin as a conventional steady state equilibrium.*

[4]See Solow (1956), pp.76-77. Solow derives the counterpart to (5) but does not discuss the case $k(0) = 0$.

2 Discussion

2.1 What is Puzzling?

2.1.1 Causality

It is customary in economics to attribute sequential causality to dynamical processes as follows:

capital input → commodity output → saving → change in the capital stock

Thus in equations (1), (2) and (4), the r.h.s. which is unconsumed output is seen as the cause that precedes the l.h.s. which is the effect. The non-null solutions (3), (6) or (8) are puzzling because they do not seem to fit this sequential reasoning. For the change in the capital stock to become positive starting from zero, there would have to be a prior change in output and capital input from initial levels of zero. However there does not seem to be any endogenous explanation for that to happen.

Equally, the time paths of capital and output in the Solow model are usually regarded as determinate. When despite the strictly concave production function and constant saving ratio more than one path emerges, one wonders if there is some hidden force *other than changes in capital input* that brings about changes in production.

Questions such as these point to a modeling problem: what we seem to have in mind is not expressed in our model.

2.1.2 Indeterminacy

Reverting to the f of Example 1 and the solution (3), note that for $k(t) > 0$, we have $\ddot{k}(t) = s^2 f(k(t)).f'(k(t)) = 2$. In general, the expression $f(k)f'(k)$

is undefined for $k = 0$ given $f(0) = 0$ and $f'(0) = +\infty$. For the non-null solution (3), $f(k).f'(k) \to 8$ as $k \to 0$, i.e., $f'(k)$ blows up to $+\infty$ faster than $f(k)$ goes to zero. This assumes that the composite function $f \circ f'$ is continuous at $k = 0$. However consider the solution $k(t) = 0$ for $t \geq 0$. In that case $\ddot{k} = 0$, hence a discontinuity is to be attributed to the composite function at $k = 0$[5]. One way by which to dispose of the problem is by imposing the condition on *economic grounds* that the value of this composite function is zero at the origin.

However that amounts to begging the question. Do we have *a priori* economic reasons to impose this condition? Instead of pushing this particular line of argument, let us pursue the question of sequential causality a bit further.

3 Discrete Time Modeling

3.1 A Discrete Example

It is easier to capture sequential reasoning in terms of discrete time analysis. There is a simple discrete time counterpart to the continuous-time growth trajectory of Example 1. Consider the following:

Example 2 *Continuing to assume that the labour force is constant and normalized to unity and there is no depreciation, the basic growth equation in discrete time is*

$$k_{t+1} = sF(k_t) + k_t, t = 0, 1, 2, ... \tag{9}$$

[5]Corresponding remarks can be made for the solution in (8) for other values of a using the appropriate n-th order derivative of $k(t)$.

Consider the special case when $F(k) = 2 + 4\sqrt{k}$ *and* $s = 1/2$*. Therefore* $k_{t+1} = (1 + \sqrt{k_t})^2$ *and so we obtain the unique solution*

$$k_t = (t + \sqrt{k_0})^2 \tag{10}$$

Therefore for $k_0 = 0$*, we obtain*

$$k_t = t^2, \ t = 0, 1, 2, ... \tag{11}$$

here as well.

Once again there is growth from a zero stock. In the present case that is not surprising given that $F(0) > 0$: capital is not essential for production. Yet at one level both examples give rise to the same kind of concern: how to interpret the production of something from nothing. (In the continuous time version of (3), $k(0) = 0$ but $k(t + h) > 0$ for all $h > 0$, however tiny). To be a bit more pointed, exactly how similar or dissimilar are the two examples?

Note that in both cases, the "something" seems to come a little *after* the application of "nothing". In the continuous-time model, initially, the instantaneous rate of production is zero but if we wait for even a tiny interval of time, there is output. In the discrete time case, this interval is fixed at one unit. Therefore one way of thinking about the problem is to view it in the context of a production lag. The simplest way to do this would seem to be in terms of the Austrian concept of "waiting": output is produced with two kinds of inputs, material inputs plus "time". The production process is like an aging process; or that is how it appears to an external observer. What happens in the interior of the process is unspecified. Clearly the process is *vertically integrated* and at the start there need not be any externally supplied fixed capital input. The production of fixed capital is part of the process -

there is a construction period that makes up the initial part of the process. From the outside, *in the initial phase output is produced by time alone.*

What happens if we compress the length of the period of production?

3.2 Arbitrary Periods of Production

Let us assume that there are two inputs, fixed capital stock k and "waiting" h. The stock has a natural propensity to grow, like a tree. This means fixed capital k and waiting h are separable. At any date t, there is a stock of fixed capital - say the quantity of timber in the tree - and this is denoted as k_t. The period lasts for a length of time h and by the end of this period the stock will have grown by an amount given by $F(k_t, h)$. Here F is a stock production function which should be distinguished from the flow production function $f(k)$ used earlier. We will refer to $[t, t+h]$ also as the production period.

There is no consumption at any intermediate point of the production period while the stock is experiencing pure growth. Consumption takes place at the end of a production period. Therefore the timing of consumption is discrete. It will be assumed that at the end of a production period a fraction $(1-s)$ of the additional output is consumed and a fraction s gets added to the stock of fixed capital to be used as input for the next period. Consider the production period $[t, t+h]$. The input stock for the next period $[t+h, t+2h]$ satisfies

$$k_{t+h} = sF(k_t, h) + k_t,\ \ t = 0, h, 2h, ... \tag{12}$$

Given k_0 the above defines the sequence $< k_t >$ for $t = 0, h, 2h, ...$

Example 3 *Assume that* $F(k, h) = 2h(h+2\sqrt{k})$. *Assume also that* $s = 1/2$.

Then

$$k_t = h(h + 2\sqrt{k_{t-h}}) + k_{t-h} = (h + \sqrt{k_{t-h}})^2 = (t + \sqrt{k_0})^2 \tag{13}$$

Again for $k_0 = 0$ *we have exactly the trajectories of Examples 1 and 2 i.e.,*

$$k_t = t^2, \;\; t = 0, h, 2h, ...$$

Thus, for arbitrary $h > 0$, this is a generalization of Example 2 where $h = 1$ was assumed. Since h is arbitrary, we can explore the link between discrete time and continuous time models by taking h to become smaller and smaller.

3.3 The Instantaneous Rate of Production

Let $g(k, h)$, $g = F/h$, denote the average rate of production per unit time. Therefore, for Example 3 we have $g(k, h) = 2h + 4\sqrt{k}$. Passing to the limit as $h \to 0$ and writing $f(k) = g(k, 0)$ we obtain the instantaneous rate of production as

$$f(k) = 4\sqrt{k}$$

which is the flow production function of Example 1.

Similarly, from the basic difference equation (13), $[(k_t - k_{t-h})/h] = h + 2\sqrt{k_{t-h}}$ and taking limits as $h \to 0$ we get exactly the basic growth equation (2) of Example 1.

3.4 Multiple Solutions

Note that the above difference equation (13) has the unique solution $k_t = t^2$ for any $h > 0$. That means Example 3 is the discrete-time counterpart of one of the solutions, the non-null solution of Example 1. Example 3 does not generate the degenerate solution $k_t = 0, t = 0, 1, 2, ..$ *ad infinitum* because it is *not* the case that if $k_{t-h} = 0$ then $k_t = 0$. To generate that solution we need a different discrete-time specification.

Let

$$F(k, h) = 4h\sqrt{k} \tag{14}$$

Then $k_t - k_{t-h} = F(k_{t-h}, h)/2 = 2h\sqrt{k_{t-h}}$; therefore, $F(0, h) = 0$ implies that the only solution for the case $k_0 = 0$ is $k_t = 0, t = 0, 1, 2, ...$ *ad infinitum.*

This shows that for $k_0 = 0$, *different* extensive form models of discrete time get compressed to a single reduced form model in continuous time, leading to multiple solutions in the latter.

The discussion of Section 3.4 seems to point to the following conclusion: given the Inada condition, the Solow model is *under-specified* for the case $k(0) = 0$.

References

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