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Auctions with endogenous quantity revisited

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Abstract

In a two-bidder procurement auction with variable quantity, Hansen (1988) shows the following. (i) The expected price at which the good is sold is lower under first-price auction (FPA) as compared to the expected price under second-price auction (SPA). (ii) The expected quantity sold under FPA is greater as compared to the expected quantity sold under SPA. We generalise his first result (by considering n bidders) and provide an alternative direct proof. Next, we show that the second result is not always true. We provide the correct version of this result and give some examples to illustrate our point.

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1 Introduction

Traditionally auction theory has analysed situations where a fixed number of units is up for sale. The symmetric, independent, private value (SIPV) model has been the benchmark model for auctions[2]. In a variable quantity auction, the amount sold is not fixed but depends on the level of the winning bid. Till date, only a handful of papers have dealt with such auctions. Hansen (1988) is a pioneering contribution in this regard. He constructs a procurement auction model where two risk neutral sellers compete for the right to sell to a market characterised by a negatively sloping demand curve $q(p)$. In this environment the quantity traded becomes endogenous. In such an auction, Hansen (1988) derives two main results. (i) The second price auction (SPA) yields a higher expected price than the first price auction (FPA)[3]. (ii) The expected quantity sold is higher with FPA. Conventional wisdom suggests that this is why most procurements take the form of FPA.[4]

In this note our objective is twofold. We first generalise (by considering n bidders) the result that SPA leads to higher expected price than FPA and provide an alternative direct proof. Next, we show that if demand $q(p)$ is concave then the expected **quantity** sold is higher with FPA. But when demand is not concave this is not necessarily true. We produce a couple of examples to illustrate our point. It may be noted that Hansen (1988)

[2]See Krishna (2002) and Milgrom (2004) for all the important results on the SIPV model.

[3]It may be noted that the SPA is outcome equivalent to the English auction and the FPA is outcome equivalent to the Dutch auction. See the appendix also.

[4]In this connection see Milgrom (1989 and 2004, pp. 135-137), who provides an elegant discussion of the Hansen results. Klemperer (2004 pp. 31) and Burguet (2000, pp. 39-40) also provide a nice exposition.

and others (like Milgrom, 1989 and Klemperer, 2004) claim that expected quantity will *always* be higher with FPA. Our exercise shows that this is simply not true.

The plan of our paper is as follows. In section 2 we provide a brief discussion of the auction theoretic terms used in this paper. In section 3 we provide the model of our exercise. In this section we also briefly discuss the basics of order statistics. Section 4 discusses the bidding equilibrium in both FPA and SPA. Section 5 gives our main results and in the last section we provide some concluding remarks.

2 Preliminaries

We now provide an explanation of some of the terms used in this paper[5]. There is seller who has a single indivisible object. The seller does not know how much any buyer would be willing to pay for it. If the seller were to know each buyer's value for the object, he could just approach the buyer who values it most. This strategy is infeasible when he does not know their values. The reason the seller holds an auction is because his information about the possible buyers is imperfect; the auction is intended to produce the best sale price in part by identifying the best bidder. A *standard auction* is where the object is sold to the highest bidder.

The terms The *value* (or reservation price) for a bidder i, denoted by v_i, is the maximum amount of money he would be willing to pay for the object. In game (auction) theory jargon, v_i is the type of bidder i.

[5]We closely follow Mathews (1995) for this appendix. Krishna (2002), Milgrom (2004) and Klemperer (2004) provide explanations of all the terms discussed here.

The symmetric, independent, private value (SIPV) model of auctions make the following assumptions.

1. *Private values* : The private information of a bidder is his own value for the object, and it does not depend on what the other bidders know.

2. *Independent types*: $v_1, v_2....v_n$ are independently distributed.

   Bidder i believes that $v_1, v_2..v_{i-1}, v_{i+1}..v_n$ are random variables to which he can attribute a joint probability distribution. Each bidder believes that the others' types are distributed independently of his own.

3. *Symmetry* : Each random variable v_i has the same distribution.

4. *Risk Neutrality* The bidders are risk neutral.

The four standard auctions are as follows.

1. *First Price auction (FPA)* : The bidders simultaneously submit sealed bids. The highest bidder wins and pays a price equal to his bid.

2. *Second Price Auction (SPA)* : The bidders simultaneously submit sealed bids. The highest bidder wins and pays a price equal to the second highest bid.

3. *English Auction (EA)* : Bids are oral. The auctioneer (the seller) starts the bidding process at some price. The bidders proclaim successively higher bids until no bidder is willing to bid higher. The bidder who submitted the final bid wins and pays a price equal to his bid. For private value auctions, EA is outcome equivalent to SPA.

4. *Dutch Auctions (DA)* : The seller starts with a very high price. Then the price declines continuously on a "wheel" until one bidder yells "stop". That bidder wins and pays the price at which the wheel stopped. The DA is outcome equivalent to the FPA.

It may be noted that in each of the above auction if there is tie at the top (i.e. if there are k highest bidders), we assume that each such bidder wins with probability $1/k$.

A *procurement auction* is one where the auctioneer is a *buyer* instead of being a seller. The bidders are the sellers. Each bidder possesses one unit of an identical object which is being offered for sale. In a standard procurement auction the *lowest bidder* wins. Here, in a FPA the lowest bidder wins and sells the object to the auctioneer (who is the buyer) at that price. Similarly, in a SPA the lowest bidder wins and sells the object to the auctioneer at the second lowest price and so on. It may be noted that procurement auctions are just standard auctions in reverse.

The auctions considered in this appendix so far are *fixed quantity* auctions - in the sense that a fixed number of units is up for sale. In contrast, in a *variable quantity auction*, the amount sold is not fixed but depends on the level of the winning bid.

Auction theory's most celebrated theorem, *The Revenue Equivalence Theorem* basically states that for *fixed quantity* SIPV auctions with risk neutral bidders the expected price (which is the same as the expected revenue), at which the good is sold, is same across all the four standard auctions. For the most general formulation of this theorem see Klemperer (2004).

We now proceed to provide the model of our exercise.

3 The Model

There is an auctioneer (buyer) with demand $q(p)$ trying to procure some amount of a product through an auction. There are n firms who manufacture the product. They submit bids (that is prices) at which they want to sell. The lowest bid wins the contract.[6] In this set up, the sellers compete for the right to sell to a market characterised by a negatively sloped demand curve $q(p)$. Each firm has a constant marginal cost c_i. The buyer can choose either a FPA or a SPA.

We list the assumptions of our model below.

Assumptions

1. Seller $i's$ MC c_i is private knowledge to the seller. Seller i knows c_i and not c_j $(j \neq i)$.

2. Independent types: $c_1....c_n$ are independently distributed. Seller i believes that

   $c_1, c_2...c_{i-1}, c_{i+1}, ...c_n$ are random variables to which he can attribute a joint probability distribution.

3. Symmetry: Each random variable $c_i \in [\alpha,\ \beta]$ has the same distribution function $F(.)$ and associated density $f(.)$. That is, each seller i believes that competitors' MCs are given by $c_j \in [\alpha,\ \beta]$ with distribution function $F(.)$ and density function $f(.)$. We assume $\alpha > 0$.

4. The auctioneer and the sellers are all risk neutral.

[6]In case of a tie the winner is selected randomly. It may be noted that such ties occur with zero probability in equilibrium.

It may be noted that most assumptions (except the fact that this is an auction with variable quantity) are very similar to the SIPV model.

Action set of seller i is $[0, \infty) \cup \{No\}$. That is, each seller can quote either a non-negative number or can say "no" which means that it drops out of the bidding process.

Strategies are bids which is a function of types (costs)

$$b_i(c_i) : [\alpha,\ \beta] \longrightarrow [0, \infty) \cup \{No\}$$

We look for a profile of strategies that constitute a Bayesian- Nash equilibrium. Before giving our main results we need to discuss some preliminaries on order statistics.

3.1 Order Statistics : some notations and preliminaries

Let $y_1, y_2..y_n$ denote a random sample of size n drawn from $F(.)$. Then $x_1 \leq x_2... \leq x_n$ where $x_i s$ are $y_i s$ arranged in increasing magnitudes, are defined to be the order statistics corresponding to the random sample $y_1, y_2....y_n$.

We would be interested in x_1 (lowest order statistic) and x_2 (second lowest order statistic). The corresponding distribution functions and density functions are $F_1(.),\ F_2(.)$ and $f_1(.),\ f_2(.)$. Note that

$$\begin{aligned} F_1(x) &= 1 - (1 - F(x))^n \text{ and } F_2(x) = 1 - (1 - F(x))^n - nF(x)(1 - F(x))^{n-1} \\ f_1(x) &= n(1 - F(x))^{n-1} f(x) \text{ and } f_2(x) = n(n-1)F(x)(1 - F(x))^{n-2} f(x) \end{aligned}$$

It may be noted that x_1 and x_2 are not independent although the underlying distributions are. We will be using the above formulae in our proofs. See the appendix of Wolfstetter (1999) or Mood, Graybill and Boess (1974, pp. 251-265) for a discussion of order statistics.

4 The Equilibrium in Auctions

4.1 Equilibrium bidding in FPA

In the FPA if firm j is the winner it gets

$$\pi_j = (b_j - c_j)\, q(b_j).$$

All the losers get zero. It follows from Hansen (1988) that there exists a symmetric Bayesian Nash equilibrium with strategies $b(c_i)$ which is strictly increasing in c_i. $b(c_i)$ solves the following differential equation (1) with the boundary condition (1a).

$$\begin{aligned} \frac{db}{dc} &= (n-1)\frac{f(c_i)}{1-F(c_i)}\left[\frac{(b(c_i)-c_i)\,q(b(c_i))}{q(b(c_i))+(b(c_i)-c_i)\,q'(b(c_i))}\right] \quad ----(1) \\ \mathit{and}\; b(\beta) &= \beta \quad ----(1a) \end{aligned}$$

It may be noted that $b(c_i) > c_i$ for all $c_i \in [\alpha, \beta)$. In equation (1), the term in the denominator $q(b(c_i)) + (b(c_i) - c_i)\, q'(b(c_i))$ is strictly positive in the relevant range. Equation (1a) suggests that the worst-off seller - the seller with highest possible cost, β - has zero expected profit. It can also be shown that $b(c_i)$ is a concave function.

We now proceed to provide a brief explanation of the above claims. We will closely follow Hansen (1988) here. A seller with cost c_i quotes b_i to maximise

$$\exp\text{ected profit} = E(\pi_i) = \Pr ob.\,(win \mid b_i)\, q(b_i)\,(b_i - c_i).$$

When there is a common bidding strategy $b(c_i)$, where $b(.)$ is strictly increasing, the probability of winning is equal to $(1 - F(c_i))^{n-1}$, the probability that the other sellers' costs are higher than his. Then

$$E(\pi_i) = (1 - F(c_i))^{n-1} q(b(c_i))(b(c_i) - c_i).$$

From the above it is clear $b(c_i) > c_i$; because otherwise expected profits are going to be non-positive. Now, fix the strategy $b(c)$, and allow each seller to choose the c he wishes to report. Then $b(c)$ is an equilibrium strategy if sellers choose honest revelation (i.e. they choose to report $c = c_i$). Under this

$$E(\pi_i) = (1 - F(c))^{n-1} q(b(c))(b(c) - c_i).$$

Taking the derivative w.r.t. c, equating this to zero (for the first order condition) and then noting that in equilibrium $c = c_i$ we get equation (1).

The prices quoted by the sellers lie between $b(\alpha)$ and β. In a FPA, the firm with the lowest cost bids the lowest price and wins the contract. The price at which the items are procured is the lowest of all bids. Therefore, the expected price in a FPA is as follows.

$$E^I(price) = \int_\alpha^\beta b(x) f_1(x)\, dx.$$

Note that $f_1(x)$ is the density function of the lowest order statistic. Similarly the expected quantity traded in a FPA is the following

$$E^I(qty.) = \int_\alpha^\beta q(b(x)) f_1(x)\, dx$$

4.2 Equilibrium bidding in SPA

It can be easily shown that in equilibrium each seller bids his MC c_i. Prices quoted by the sellers in the SPA will lie between α and β. In this auction the firm with the lowest cost wins the contract but the price at which the good is sold is equal to the second lowest cost. The expected price in the SPA is therefore the expected value of the second lowest order statistic.

$$E^{II}(price) = \int_{\alpha}^{\beta} x f_2(x)\, dx.$$

Note that $f_2(x)$ is the density of the second lowest order statistic. Similarly the expected quantity traded in a SPA is the following

$$E^{II}(qty.) = \int_{\alpha}^{\beta} q(x)\, f_2(x)\, dx$$

5 Expected prices and quantities

We now provide our main results and their proofs.

Proposition 1 *The expected price at which the goods are sold is strictly lower under FPA as compared to the expected price under SPA. That is,*

$$\int_{\alpha}^{\beta} b(x)\, f_1(x)\, dx < \int_{\alpha}^{\beta} x f_2(x)\, dx.$$

Proof We have

$$\int_{\alpha}^{\beta} b(x) f_1(x) dx = \int_{\alpha}^{\beta} nb(x)\left(1 - F(x)\right)^{n-1} f(x) dx$$

Note that $f(x)dx = F'(x)dx$.

Let $u(x) = nb(x)\left(1 - F(x)\right)^{n-1}$. Therefore

$$u'(x) = n\left[\left(1 - F(x)\right)^{n-1} b'(x) - (n-1)\, b(x)\left(1 - F(x)\right)^{n-2} f(x)\right]$$

Integrating by parts we get

$$\int_{\alpha}^{\beta} b(x)\, f_1(x)\, dx = \int_{\alpha}^{\beta} u(x) F'(x)\, dx = \left[u(x)F(x)\right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} F(x) u'(x) dx$$

Therefore

$$\begin{aligned}
&\int_{\alpha}^{\beta} b(x)\, f_1(x)\, dx \\
= \; & \left[nb(x)\,(1-F(x))^{n-1}\,F(x)\right]_{\alpha}^{\beta} \\
& -\int_{\alpha}^{\beta} F(x)n\left((1-F(x))^{n-1}\,b'(x)-(n-1)\,b(x)\,(1-F(x))^{n-2}\,f(x)\right)dx---(2)
\end{aligned}$$

Note that

$$\left[nb(x)\,(1-F(x))^{n-1}\,F(x)\right]_{\alpha}^{\beta}=0 \text{ as } F(\alpha)=0 \text{ and } F(\beta)=1.$$

Also substituting the value of $b'(x)$ from (1) into (2) we get

$$\begin{aligned}
&\int_{\alpha}^{\beta} b(x)\, f_1(x)\, dx \\
= \; & \int_{\alpha}^{\beta} F(x)n\left[\begin{array}{c} (n-1)\,b(x)\,(1-F(x))^{n-2}\,f(x) \\ -(1-F(x))^{n-1}\,(n-1)\frac{f(x)}{1-F(x)}\left[\frac{(b(x)-x)q(b(x))}{q(b(x))+(b(x)-x)q'(b(x))}\right]\end{array}\right]dx \\
= \; & \int_{\alpha}^{\beta} n(n-1)\,(1-F(x))^{n-2}\,f(x)F(x)\left(b(x)-\frac{(b(x)-x)\,q(b(x))}{q(b(x))+(b(x)-x)\,q'(b(x))}\right)dx \\
= \; & \int_{\alpha}^{\beta} f_2(x)\left(b(x)-\frac{(b(x)-x)\,q(b(x))}{q(b(x))+(b(x)-x)\,q'(b(x))}\right)dx----(3)
\end{aligned}$$

Equation (3) follows because $f_2(x)=n(n-1)\,(1-F(x))^{n-2}\,f(x)F(x)$ (see the discussion on order statistics). Note that $E^{II}\,(price)=\int_{\alpha}^{\beta} xf_2(x)dx$. Therefore from (3) we get that

$$E^{II}\,(price)-E^{I}\,(price)=\int_{\alpha}^{\beta} f_2(x)\left(x-b(x)+\frac{(b(x)-x)\,q(b(x))}{q(b(x))+(b(x)-x)\,q'(b(x))}\right)dx----(4)$$

Note that

$$x-b(x)+\frac{(b(x)-x)\,q(b(x))}{q(b(x))+(b(x)-x)\,q'(b(x))}=\frac{-(b(x)-x)^2\,q'(b(x))}{q(b(x))+(b(x)-x)\,q'(b(x))}----(5)$$

Now $b(x) > x$ for all $x \in [\alpha, \beta)$, $q'(.) < 0$ and $q(b(x)) + (b(x) - x) q'(b(x)) > 0$. Therefore from (5) we get that

$$x - b(x) + \frac{(b(x) - x)\, q(b(x))}{q(b(x)) + (b(x) - x)\, q'(b(x))} > 0 \text{ for all } x \in [\alpha, \beta).$$

Using the above inequality in (4) we conclude that $E^{I}(price) < E^{II}(price)$.■

Comment In the SIPV auction model where one fixed item is up for sale, the expected price at which the good is sold is same across FPA and SPA (the revenue equivalence theorem). Note that for auctions with one fixed item, expected price is same as expected revenue. In variable quantity auction this is not so. Proposition 1 shows that in procurement auctions with variable quantity, the *expected prices* are not same across the two auctions. Unlike the revenue equivalence theorem, proposition 1 does not say anything on expected revenues.

Also note that the result ($E^{I}(price) < E^{II}(price)$) holds irrespective of the curvature of $q(p)$. We now proceed to the next main result, which crucially depends on the shape of $q(p)$.

Proposition 2 *If $q(p)$ is concave then the expected quantity sold is higher under FPA as compared to SPA.*

Proof We have

$$\int_{\alpha}^{\beta} q(b(x))\, f_1(x) dx = \int_{\alpha}^{\beta} nq(b(x))\, (1 - F(x))^{n-1} f(x) dx$$

Let $g(x) = nq(b(x))(1 - F(x))^{n-1}$. Therefore

$$g'(x) = n\left[(1 - F(x))^{n-1} q'(b(x))\, b'(x) - (n-1)\, q(b(x))\, (1 - F(x))^{n-2} f(x)\right]$$

Noting that $f(x) = F'(x)$ and integrating by parts we get

$$\int_{\alpha}^{\beta} q(b(x)) f_1(x)\, dx = \int_{\alpha}^{\beta} g(x)F'(x)\, dx = [g(x)F(x)]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} F(x)g'(x)dx$$

Clearly $[g(x)F(x)]_{\alpha}^{\beta} = 0$. Therefore

$$\int_{\alpha}^{\beta} q(b(x)) f_1(x)\, dx$$
$$= \int_{\alpha}^{\beta} nF(x)\left[(n-1)\, q(b(x))(1-F(x))^{n-2} f(x) - (1-F(x))^{n-1} q'(b(x))\, b'(x)\right] dx$$

Substituting the value of $b'(x)$ from (1) and cancelling and rearranging terms (as in the proof of proposition 1) we get

$$\int_{\alpha}^{\beta} q(b(x)) f_1(x)\, dx$$
$$= \int_{\alpha}^{\beta} f_2(x)\left[q(b(x)) - q'(b(x))\frac{(b(x)-x)\, q(b(x))}{q(b(x)) + (b(x)-x)\, q'(b(x))}\right] dx ----(6)$$

Note that $E^{II}(qty.) = \int_{\alpha}^{\beta} q(x) f_2(x)\, dx$. Therefore from (6) we get

$$E^{I}(qty.) - E^{II}(qty.)$$
$$= \int_{\alpha}^{\beta} f_2(x)\left[q(b(x)) - q'(b(x))\frac{(b(x)-x)\, q(b(x))}{q(b(x)) + (b(x)-x)\, q'(b(x))} - q(x)\right] dx$$
$$= \int_{\alpha}^{\beta} f_2(x)\left[\frac{q(b(x))\,[q(b(x)) - q(x)] - q(x)(b(x)-x)\, q'(b(x))}{q(b(x)) + (b(x)-x)\, q'(b(x))}\right] dx --- (7).$$

Note that $b(x) > x$ for all $x \in [\alpha, \beta)$. By using the mean value theorem we get that

$$q(b(x)) - q(x) = (b(x)-x)\, q'(y) \text{ for some } y \in (x, b(x)) ----(8)$$

Substituting the value of $q(b(x)) - q(x)$ from (8) in (7) we get

$$E^{I}(qty.) - E^{II}(qty.) = \int_{\alpha}^{\beta} f_2(x)\left[\frac{(b(x)-x)\,[q(b(x))\, q'(y) - q(x)\, q'(b(x))]}{q(b(x)) + (b(x)-x)\, q'(b(x))}\right] dx ----(9)$$

Note that the denominator of the RHS of (9) is positive. Since $b(x) > x$ for all $x \in [\alpha, \beta)$, we have $q(b(x)) < q(x)$ for all $x \in [\alpha, \beta)$. Since $q(.)$ is concave $q'(b(x)) \leq q'(y)$. Also $q'(.) < 0$. Combining all these we get that

$$\frac{(b(x) - x)\left[q(b(x))\, q'(y) - q(x)\, q'(b(x))\right]}{q(b(x)) + (b(x) - x)\, q'(b(x))} > 0, \text{ for all } x \in [\alpha, \beta).$$

Using the above inequality in (9) we get that $E^{I}(qty.) > E^{II}(qty.)$.■

Comment If $q(p)$ is not concave then $E^{I}(qty.)$ may not be greater than $E^{II}(qty.)$. We produce two examples to illustrate our point.

Example 1 Let $q(p) = 1/p$. Note that in this particular example we have

$$q(b(x)) = \frac{1}{b(x)},\ q'(b(x)) = -\frac{1}{(b(x))^2} \text{ and } q(x) = \frac{1}{x}.$$

Then clearly

$$q(b(x))\left[q(b(x)) - q(x)\right] - q(x)(b(x) - x)\, q'(b(x)) = 0$$

and using (7)[7] we conclude that $E^{I}(qty.) = E^{II}(qty.)$ in this case.

Example 2 To show that $E^{I}(qty.)$ can be strictly lower than $E^{II}(qty.)$ we produce the following example. Let $q(p) = 1/\sqrt{p}$. In this example we have

$$q(b(x)) = \frac{1}{\sqrt{b(x)}},\ q'(b(x)) = -\frac{1}{2(b(x))^{\frac{3}{2}}} \text{ and } q(x) = \frac{1}{\sqrt{x}}.$$

[7] The denominator of (7) is always positive.

Then clearly

$$\begin{aligned}
& q\left(b\left(x\right)\right)\left[q\left(b\left(x\right)\right)-q\left(x\right)\right]-q\left(x\right)\left(b\left(x\right)-x\right)q'\left(b\left(x\right)\right) \\
= \; & -\frac{1}{2}\frac{-2\sqrt{b\left(x\right)x}+b\left(x\right)+x}{\sqrt{x}\left(b\left(x\right)\right)^{\frac{3}{2}}}=-\frac{1}{2}\frac{\left(\sqrt{b\left(x\right)}-\sqrt{x}\right)^{2}}{\sqrt{x}\left(b\left(x\right)\right)^{\frac{3}{2}}}<0 \text{ for all } x\in\left[\alpha,\beta\right) \\
\Longrightarrow \; & E^{I}\left(qty.\right)<E^{II}\left(qty.\right) \;\; \text{(from (7))}.
\end{aligned}$$

The interesting point to note is that the conclusions of the above two examples hold true for any $F\left(c\right)$ (the distribution function of costs). Also, we did not need to compute the bidding strategies $b\left(x\right)$ explicitly in either example to arrive at our conclusions.

6 Conclusion

In this note we have shown that in procurement auctions with variable quantity the expected price at which the goods are traded is *always* lower under FPA as compared to SPA. However, whether expected quantity of the goods sold is higher under FPA depends crucially on the curvature of $q\left(.\right)$. If $q\left(.\right)$ is concave then the expected quantity sold is strictly higher under FPA. However, if $q\left(.\right)$ is not concave then the expected quantity sold may be equal or may even be strictly lower under FPA. We demonstrated this possibility with two examples.

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